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By Pearn P. Niiler
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Journal of Marine Research, Sears Foundation for Marine Research, Yale University
PO Box 208118, New Haven, CT 06520-8118 USA
(203) 432-3154 (203) 432-5872 jmr@yale.edu www.journalofmarinereseach.org

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ABSTRACT

A model is given that describes the local response of the upper ocean to an imposed surface wind stress and heat flux. The stress, τ_0, drives mean motions in a vertically homogeneous column of depth, h, and the interaction of this stress with the wave-induced vertical shear of surface currents produces a portion of turbulent kinetic energy to maintain a homogeneous column. Additional turbulent energy is generated at the base of the mixed layer by the interaction of the turbulent entrainment stress with the shear of the mean motion. Erosion of a stably stratified, quiescent layer in the ocean is initially characterized by a rapid deepening, qualitatively predicted by Pollard, Rhines, and Thompson (1973), and a subsequent slow erosion, as parameterized by Kraus and Turner (1967), follows. If the mixed-layer depth over a buoyancy frequency layer, N, at the onset of a strong wind is less than

$$h_{\min} \simeq (2 \sim 1.8) \times (\tau_0/\varrho_0 N f)^{1/2},$$

within half a pendulum day of π/f, the inertial motions provide the turbulent energy to mix the layer to $h_{\min}$. In the absence of heating, and in time scales $f^{-1} << t$, and $t < N^{-1}$, the erosion process quantitatively follows

$$h \sim (12\ \mathrm{m}_0)^{1/3} (\tau_0/\varrho_0 N^2)^{1/3} (tN)^{1/3},$$

where m_0 is the fraction of the turbulent energy generated in the wave zone that is available for mixing. It is shown that inertial motions play an important role in mixing in an *intermediate* time scale $N^{-1} < t < f^{-1}$. When persistent surface heating of $\dot{Q}_0$ is present, the slow erosion process is arrested at a depth

$$h_{\max} \sim 2\ \mathrm{m}_0 \left(\frac{\tau_0}{\varrho_0}\right)^{3/2} C_p \varrho_0{}^2/\alpha g \dot{Q}_0,$$

where C_p is the specific heat of sea water, α is the thermal expansion coefficient, and g is gravitational acceleration. Recent interpretations of observations in the ocean are discussed in light of this model.

1. Introduction

The adjustment of the temperature and velocity structure of the surface layers of the ocean to variable surface fluxes of heat and momentum has been the subject of a large number of studies since Ekman's (1905) treatise, "On the influence of earth's rotation on ocean-currents." Two recent papers quite adequately summarize

1. Nova University, Fort Lauderdale, Florida, 33004, U.S.A. *Present address:* Department of Oceanography, Oregon State University, Corvallis, Oregon, 91331.

the theory and the observations of the deepening of homogeneous, wind-mixed layers. A study by Pollard, Rhines, and Thompson (1973) (henceforth, P.R.T.) points out that the wind generates inertial motions within the mixed layer and explores a model of the erosion of a stably stratified ocean which is driven by the available turbulent energy within these motions. The authors extend the lines of investigation initiated by Geisler and Kraus (1969) and demonstrate how partition of available turbulent energy can occur in a time scale of a few pendulum hours. Denman and Miyake's (1973) study represents a thorough comparison of ocean station data with Kraus and Turner's (1967) and Miropolskiy's (1970) model of wind-driven deepening. The latter calculations point out the role of persistent surface wind-stirring in supplying the mechanical energy for mixing the surface water mass over a week's time scale.

Observations indicate that both processes must be important. Energetic inertial motions within the mixed layer are observed to be directly related to the surface wind-stress history (Pollard and Millard, 1970), and over a two-week stormy period, a continual erosion of the seasonal thermocline is a distinguishable phenomenon (Denman and Miyake, 1973). A three-layer model of ocean surface dynamics in which both processes are parameterized is discussed herein. The model is not unlike that proposed by Geisler and Kraus (1969); however, the production rate of turbulent kinetic energy from the shear of the mean motion in the entrainment zone is computed in a self-consistent manner. The model is not unlike that proposed by P.R.T.; however, the production rate of turbulent kinetic energy from the wave zone near the ocean surface is included. A variety of initial value problems are examined and is shown to model both inertial motion generation and persistent deepening in a manner suggested by the simplest interpretation of the observations. The surface production rate ($\vec{\tau}_0 \cdot U_0^*$ energetics) is shown to dominate the turbulent energy production rate from the entrainment interface for both small ($t < N^{-1}$) and large ($t > f^{-1}$) times. While observational evidence over a month's time tends to bear out the notions introduced, greater care must be exercised in extending this model to longer-period development without considerations of horizontal inhomogeneities within the ocean.

2. A three-layer model of a mixed layer

The assumptions about the vertical structure of the mixed layer introduced herein are based on observations of the structure of the mixed layer rather than on a derivation of the dynamics from first principles. A vertical shape of the "mean" temperature and velocity fields is assumed, and the turbulent fluxes of momentum and heat are then constructed, which are consistent with the conservation equations. These assumptions have been stated with varying degrees of emphasis by the various authors to whom reference is made throughout this paper. The first ingredient is the observation that the temperature (and salinity) of a column of water directly beneath

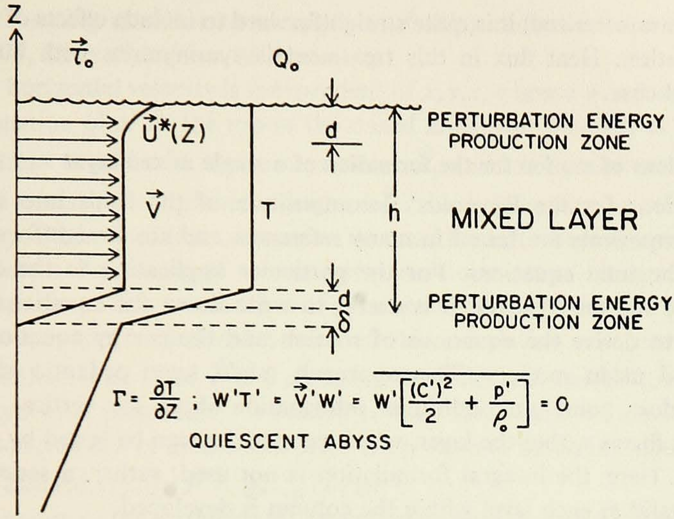

Figure 1. Schematic model of the vertical structure of temperature, velocity, and turbulent flux boundary conditions.

a windblown ocean surface is vertically uniform. The second is that, within such a column and below a wave-zone, the horizontal currents are independent of depth. Two vertical shear layers are postulated: the near-surface zone which is wave-driven, and a second vertical shear zone which is to be found near the bottom of the mixed layer. In the wave-generated shear zone, the vertical fluxes of heat and horizontal momentum are constant. The interface at the bottom of the mixed layer is an entrainment zone in which there is a finite vertical transport of heat and horizontal momentum into the top, while none leaks out from the bottom. This latter momentum flux is used to accelerate the horizontal flow below the interface to a value within the mixed layer, and the vertical heat flux is used to raise the temperature of the mixed layer from the value at the top of the seasonal thermocline to the homogeneous value within the mixed layer. The gross features of this three-layer model are sketched in Figure 1. The model is not novel, however this calculation shows how the somewhat separate points of view which have been espoused in the most recent work on the theory of formation of a local mixed layer and the onset of inertial motions within this layer during the passing of a strong open-ocean storm can be unified.

Within the context of this particular model, deepening events can be described. The sequence of events during a strong heating cycle, such as the formation of a seasonal thermocline, can be described as the formation of a sequence of progressively warmer mixed layers that occur during the daily heating cycle over cooler and isolated deeper layers. The erosion process of a sequence of shallow layers into a smooth gradient has not been parameterized, whence this model stores the history of the heating, cooling, and wind cycles in the step-like structure of the temperature field within the column near the ocean surface. Although only effects of heating and cooling are parameterized, it is quite straightforward to include effects of evaporation and precipitation. Heat flux in this treatment is synonymous with buoyancy flux in the general case.

3. The equations of motion for the formation of a single mixed layer

The equations for the Reynolds' decomposition of the fields into a mean and turbulent components are found in many references and are straightforward derivations from the total equations. For the particular application to the dynamics of this model of the mixed layer, it is useful to write down the equations again, and step-by-step to derive the equations of motion and the energy equations for both turbulent and mean motions. This approach might seem pedantic at first sight; however, it does point out additional information about the vertical structure of the turbulent fluxes within the layer, which presumably can be tested by experiments in the ocean. Here, the integral formulation is not used; rather, a series of recipes which hold valid at each level within the column is developed.

Let a field variable $G = g(z,t) + G'(\vec{x},z,t)$ have the ensemble mean $\bar{g} = \bar{G}$, whence G' is the deviation from the mean. While G' represents the collective effects of turbulence and motions which vary more rapidly than g, and these can have horizontal variations, it is assumed that the correlations among these deviations are independent of the horizontal position $\vec{x}$. Clearly, the atmospheric forcing which leads to the formation of the mixed layer is nonuniform in $\vec{x}$; it is assumed that these horizontal scales are larger than the local scales which lead to the principal correlations in G' or turbulence in the water. In principle, the slow divergences and convergences of turbulent fluxes can also be parameterized; however, at the moment, attention is focused on the local problem.

The equations for the homogeneous column are

$$\frac{\partial \vec{v}}{\partial t} + \vec{f} \times \vec{v} = -\frac{\partial}{\partial z}\left[\overline{\vec{V}'W'}\right] - \frac{\vec{F}}{\varrho_0}, \tag{1}$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial z}\left[\overline{W'T'}\right] = \dot{q}(z,t)/C_p\varrho_0. \tag{2}$$

Equations (1) and (2) apply to motion *below* the constant stress layer at the mean ocean surface and *above* the sharp entrainment interface at the bottom of the mixed layer. This distinction is important since the assumption is that, immediately below the entrainment zone the turbulent vertical transfer of heat and momentum, $-\overline{W'T'}$ and $-\overline{W'\vec{V}'}$, vanish, while they take on finite values just above the zone. In Figure 1, the surface and entrainment layers are shown to be of thickness d, in which T and $\vec{v}$ are functions of z. The derivation is carried out for finite (d/h), and then terms of this order are neglected. The usual hydrodynamical notation is used, in which

$\dot{q}(z,t)$ is the heating rate due to absorption of solar radiation at a depth z (vide Denman, 1973, for a discussion of the latter).

In (1), the horizontal velocity is independent of x,y,z, whence $w = 0$ and $-\varrho_0\overline{\vec{V}'W'}$ is a linear function of z. At the top of the mixed layer, this quantity is equal to the horizontal stress transmitted to the mean flow, $\vec{\tau}_0$, and at the bottom it is equal to $\varrho_0\vec{v}\partial h/\partial t$, the rate at which quiescent fluid is impulsively brought to a velocity $\vec{v}$. With the deepening of the mixed layer, there can be an additional stress, due to momentum transfer by internal waves, and a subsequent development of a mean flow below the bottom interface. This transfer is not parameterized directly at the present time in order that a succinct relationship between this calculation and previous models can be established. The force $\vec{F}$ is a damping force for the inertial motions within the mixed layer and, presumably, is related to the "dissipation" of mean motions as well as the radiation flux of momentum from the bottom of the mixed layer. Following Pollard and Millard (1970), $\vec{F} = \varrho_0 C_D |\vec{v}|\vec{v}h^{-1}$.

The temperature of the mixed layer can vary with depth on a space scale on which the daily absorption of short-wave solar energy takes place within the water column. However, as e^{-1}, extinction of the absorption occurs within the first two or three meters of the ocean surface (Denman and Miyake, 1973), and since this is also a region of intense wave action, it can be safely assumed that the daily heating rate acts as a surface input. Denman (1973) has made a careful examination of the equivalent radiatively produced increase of potential energy and temperature in the mixed layer at the North Pacific Station PAPA. He has concluded that when wind speeds exceed 6 m/sec, such direct deviations of the temperature that are produced by local absorption are below the noise level of other smallscale features within the layer. Hence, for the purpose of the present calculations, $\dot{q} = 0$, and the turbulent heat flux $-C_p\varrho_0\overline{W'T'}$ is also taken to be a linear function of z, with the value $\dot{Q}_0 = \dot{Q}_R - \dot{Q}_B + \dot{Q}_S - \dot{Q}_E$ at the ocean surface, and $-C_p\varrho_0\overline{W'T'}(-h)$ at the bottom of the mixed layer. The subscripts R, B, S, and E represent the incoming short-wave radiative flux, the long-wave black body back radiation to the atmosphere, and the sensible and latent heat transfer from the sea surface, respectively. (Vide Pond, 1972, and Warren, 1972, for a review on parameterization of these fluxes as a function of cloudiness, sea-surface and air temperatures, relative humidity, and wind speeds).

As the mixed layer erodes away on the stable temperature structure below, the turbulent heat flux at the bottom is used to increase the temperature from $T(z = -h - \delta)$ to $T(z = -h)$. The heat flux required for this entrainment is set equal to $\varrho_0 C_p \partial h/\partial t \times [T(z = -h) - T(z = -h - \delta)]$. (Vide Denman, 1973, for a derivation). Consider that at time $t = 0$, there exists a linearly stratified water column with a surface temperature T_0 and a temperature gradient Γ. The temperature T in (2) is taken as the deviation from T_0, whence $T(z = -h) - T(z = -h - \delta) = T + \Gamma h$. (This is the same notation as that adopted by P.R.T., while an arbitrary $T(z, t = 0)$

can also be used). It now follows that if all the surface heat flux goes into heating the column, $-\overline{W'T'}(z=-h) = \partial h/\partial t\,[T+\Gamma h]$.

The equations of motion are now written as

$$h\frac{\partial \vec{v}}{\partial t} + \vec{f}\times\vec{v}h = -C_D|\vec{v}|\vec{v} - \vec{v}\frac{\partial h}{\partial t}, \tag{3}$$

$$h\frac{\partial T}{\partial t} + (T+\Gamma h)\frac{\partial h}{\partial t} = +\frac{\dot{Q}_0}{C_p\varrho_0}, \tag{4}$$

and the turbulent vertical fluxes of momentum and heat are

$$-\varrho_0\overline{W'\vec{V}'} = \vec{\tau}_0 + \frac{z}{h}\left(\vec{\tau}_0 - \varrho_0\vec{v}\frac{\partial h}{\partial t}\right), \tag{5}$$

$$-\varrho_0 C_p\overline{W'T'} = \dot{Q}_0 + \frac{z}{h}\left[\dot{Q}_0 - C_p\varrho_0(T+\Gamma h)\frac{\partial h}{\partial t}\right]. \tag{6}$$

This representation does not use a vertical integration of the momentum and heat storage equations, but a point-by-point parameterization of the fields that presumably occur in a mixed layer that moves as a "slab" and in which the temperature is independent of depth. Note that $h(t)$ is not constrained by (3) and (4), whence to close the system, a consideration of the perturbation kinetic energy flux that is directly related to $\overline{W'T'}$ is now carried out in detail.

4. The kinetic energy equation for the perturbation within the mixed layer

The equations for the general pertubation field are given by

$$\left.\begin{aligned}&\frac{\partial \vec{V}'}{\partial t} + f\times\vec{V}' + \frac{1}{\varrho_0}\nabla P' + W'\frac{\partial \vec{U}}{\partial z} + \vec{U}\cdot\nabla\vec{V}'\\ &+\nabla\cdot\vec{V}'\vec{V}' - \frac{\partial}{\partial z}\overline{W'\vec{V}'} = \hat{k}\frac{\alpha g}{\varrho_0}T' + \nu\nabla^2\vec{V}',\end{aligned}\right\} \tag{7}$$

$$\nabla\cdot\vec{V}' = 0, \tag{8}$$

where the mean horizontal flow $\vec{U}$ is a function of z alone (recall the shear layers discussed in Section 2; we use the notation $\vec{v}$ when the mean velocity is depth-dependent). The energy equation for the slow evolution of the perturbation field is obtained by multiplying (7) by $\vec{V}'$, a subsequent use of equation (8). There is to be horizontal homogeneity of the correlations in the perturbation fields in $\vec{x}$, whence for $C'^2 = \vec{V}\cdot\vec{V}'$,

$$\frac{\partial}{\partial t}\left(\frac{\overline{C'^2}}{2}\right) + \frac{\partial}{\partial z}\left[\overline{W'\left(\frac{P'}{\varrho_0} + \frac{C'^2}{2}\right)}\right] + \overline{W'\vec{V}'} \cdot \frac{\partial \vec{U}}{\partial z} = \frac{\alpha g}{\varrho_0}\overline{W'T'} - \overline{\upsilon \nabla \vec{V}' \cdot \nabla \vec{V}'}. \tag{9}$$

It is possible to estimate (from direct measurements) the order of magnitude of four of the five terms in the perturbation kinetic energy equation within the mixed layer, while keeping in mind the assumptions that have led to the development of equations (5) and (6) for $\overline{\vec{V}'W'}$ and $\overline{W'T'}$. Table 1 is a tabulation of this calculation as well as a list of magnitudes of the physical constants used in this calculation.

During a heating cycle, $\overline{W'T'}$ is negative in the entire range of $-h < z < 0$ as it takes on a negative value at $z = 0$, $z = -h$, and in $-h < z < 0$ it is a linear function of z. Hence both terms on the righthand side of (9) are of the same negative sign ($\overline{\upsilon \nabla \vec{V}' \cdot \nabla \vec{V}'}$) is always a positive dissipative term). It is obvious that the first term on the left-hand side is negligible; however, it is difficult to estimate from existing data whether significant production of kinetic energy results from interaction of the turbulence with the mean flow. Pollard and Millard's (1970) data of the horizontal velocity field within the mixed layer is more suited for demonstrating that the mixed layer moves like a slab rather than for estimating the shear across this layer. In this

Table 1. Balance within the perturbation K.E.equation.

Pertrubation Energy Flux	Estimator	Order of Magnitude in cm²/sec³	Reference
$\frac{\partial}{\partial t}\frac{1}{2}\overline{C'^2}$	$\frac{3\pi}{T}U^{*2}$	3×10^{-4}	Phillips (1966)
$\overline{\vec{V}'W'} \cdot \frac{\partial \vec{U}}{\partial z}$	$\frac{\tau_0(\Delta U)}{h}$	$3 \times \Delta U(\text{cm/sec})^{-1} \times 10^{-4}$	Equation (5)
$\alpha g \frac{\overline{W'T'}}{\varrho_0}$	$\frac{\alpha g}{\varrho_0^2 C_p}\dot{Q}_{max}$	4×10^{-3}	Equation (6) and Denman and Miyake (1973)
$\overline{\upsilon \nabla V' \cdot \nabla V'}$	Shape of turbulent K.E. spectra	$6 - 1 \times 10^{-3}$	Grant, *et al* (1968)

Parameters:

$T = 1$ day $= 10^5$ sec

$U^{*2} = (\tau_0/\varrho_0) = 3$ cm²/sec²

$h = 100$ m

ΔU = vertical contrast of horizontal velocity within mixed layer

$\frac{\alpha g}{\varrho_0} = 2 \times 10^{-1}$ cm/sec²°C

$\varrho_0 C_p = 1$ cal/cm³

$\dot{Q}_{max} = 2 \times 10^{-2}$ cal/cm²sec model, $\partial\vec{U}/\partial z \equiv 0$ within the mixed layer; however, if a $\Delta U \sim$ 10 cm/sec difference really does exist across the mixed layer, the production rate from mean flow can be important.

Within the confines of this model, the fluctuation energy equation within the mixed layer, with $\varepsilon \equiv \overline{v\nabla\vec{V}' \cdot \nabla\vec{V}'}$, is

$$\frac{\partial}{\partial z}\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]+\varepsilon=\frac{\alpha g}{\varrho_0}\overline{W'T'}. \tag{10}$$

Since the vertical flux of horizontal momentum due to inertial waves was not taken into account explicitly below the layer, let the vertical energy flux also vanish below the mixed layer, $\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\,(z=-h-\delta)-0$. However, this is not the case at $z=-h$, the interface between the quiescent and moving fluid, and near the surface $z=0$.

The mean flow is strongly sheared near $z=-h$; near $z=0$ an additional wind-wave driven component of flow $\vec{U}^*(z)$ is postulated. The energy balance in these interface layers, of vertical scale d, is dominated by dissipation, ε, and production rate, $\overline{\vec{V}'W'}\cdot\partial\vec{U}/\partial z$, over the vertical potential energy flux, whence, on time scales of a day, the balance within these shallow shear layers is

$$\frac{\partial}{\partial z}\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]+\overline{W'\vec{V}'}\cdot\frac{\partial\vec{U}}{\partial z}+\varepsilon=0; \qquad \begin{array}{l}0>z>-d,\\ -h+d<z<-h-\delta.\end{array} \tag{11}$$

The total difference of the vertical energy transport across the mixed layer is obtained by integrating (10) from $z=-h+d$ to $z=-d$. Since the vertical energy flux has a *source* in the active production layers at $z\to 0$ and $z\to -h$ and the flux continues smoothly and passively into the interior, the interior values can be obtained from integrating (11) across the interfaces. From (10) we obtain

$$\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]\Bigg|_{-h+d}^{-d}+\int_{-h+d}^{-d}\varepsilon dz=\int_{-h+d}^{-d}\frac{\alpha g}{\varrho_0}\overline{W'T'}\,dz, \tag{12}$$

and from integrating (11) across the interfaces,

$$\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]\Bigg|_{-d}^{0}=-\int_{-d}^{0}\overline{W'\vec{V}'}\cdot\frac{\partial\vec{U}^*}{\partial z}-\int_{-d}^{0}\varepsilon dz, \tag{13}$$

$$\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]\Bigg|_{-h-\delta}^{-h+d}=-\int_{-h-\delta}^{-h+d}\overline{W'\vec{V}'}\cdot\frac{\partial\vec{U}}{\partial z}-\int_{-h-\delta}^{-h+d}\varepsilon dz. \tag{14}$$

The horizontal Reynold's shear stress is postulated to be constant within the surface layer, whence the expression for $\overline{\vec{V}'W'}$ is obtained from equation (5) at $z = 0$. With terms of order (d/h) neglected, it should be noted that $\int_{-d}^{0} \overline{W'\vec{V}'} \cdot \frac{\partial \vec{U}^*}{\partial z} \simeq - \frac{\vec{\tau}_0}{\varrho_0} \cdot \vec{U}^*(d)$. More care is to be taken in integrating (14) through the lower boundary. Since $\vec{U}(z)$ is to be (approximately) a linear function of the integration variable, z', within the lower boundary, $\vec{U}(z) = \vec{v}\left[\frac{z+h+\delta}{\delta+d}\right] = \vec{v}\frac{z'}{(\delta+d)}$. The turbulent stress is not constant through this layer but is given by integrating the momentum equations from a level $z = -h-\delta$ where the stress vanishes, whence $\overline{\vec{V}'W'} = -\frac{z'}{\delta+d}\vec{v}\frac{\partial h}{\partial t}\left[1+0\left(\frac{d}{h}\right)\right]$. Hence, the contribution from the lower boundary is $\int_{0}^{\delta+d} \overline{W'\vec{V}'} \cdot \frac{\partial \vec{U}}{\partial z} dz' = -\frac{1}{2}\vec{v}\cdot\vec{v}\frac{\partial h}{\partial t}$ $\left[1+0\left(\frac{d}{h}\right)\right]$.

Letting $\vec{U}^*(0) = \vec{U}_0^*$, and substituting the expressions for the turbulent energy fluxes from (13) and (14) into (12), we obtain

$$\left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]\Bigg|^0 + \int_{-h}^{0} \varepsilon dz - \frac{\vec{\tau}_0}{\varrho_0}\cdot\vec{U}_0^* - \frac{\vec{v}\cdot\vec{v}}{2}\frac{\partial h}{\partial t} = \frac{\alpha g}{\varrho_0}\int_{-h}^{0}\overline{W'T'}\,dz. \tag{15}$$

The physical interpretation of (15) can be seen to be the following. There is a vertical kinetic energy flux into the mixed layer at $z = 0$, $\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\Big|^0$, which is primarily produced by the rate of work done by atmospheric pressure perturbations within the atmospheric turbulent boundary layer over the waves. Within the wave layer, there is an additional source of perturbation energy due to kinetic energy release from the turbulent wind-wave driven layer, and a third source is the kinetic energy release associated with the shear of the mean flow at the deeper interface. The wind-wave source is that which was originally proposed by Kraus and Turner (1967) as the source for mixing energy within the homogeneous layer and, subsequently, has been shown by Denman and Miyake (1973) to be relevant to the ten- to twenty-day development of the mixed layer. The entrainment zone production is the source advocated by P.R.T. and has been shown to be important in the initial half-day deepening of the layer. It simply follows that the excess perturbation energy flux over the rate of work done in dissipation is available for doing work against the gravitational field. In this model, no leakage is allowed from the bottom of the column. It can be shown that $-\frac{\alpha g}{\varrho_0}\int_{-h}^{0}\overline{W'T'}\,dz$ is the rate at which potential energy of the mean flow is changing, and using equation (6) we obtain

$$-\frac{\alpha g}{\varrho_0}\int_{-h}^{0}\overline{W'T'}\,dz = \frac{\alpha g}{\varrho_0}\left[\frac{h\dot{Q}_0}{2\varrho_0 C_p} + \frac{h}{2}(T+\Gamma h)\frac{\partial h}{\partial t}\right]. \tag{16}$$

To obtain the total energy equation, we multiply equation (3) by $\vec{v}$, substitute for $(\dot{Q}_0/\varrho_0 C_p)$ from (4) into (16), eliminate $\frac{\alpha g}{\varrho_0}\int_{-h}^{0}\overline{W'T'}\,dz$ from (15), and add the perturbation kinetic energy equation to the mean kinetic energy equation.

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left[h\frac{\vec{v}\cdot\vec{v}}{2}\right] + \frac{\alpha g}{\varrho_0}\left[\frac{h^2}{2}\frac{\partial T}{\partial t} + h(T+\Gamma h)\frac{\partial h}{\partial t}\right] - \frac{\vec{\tau}_0}{\varrho_0}\cdot\vec{v} = \\ &\frac{\vec{\tau}_0}{\varrho_0}\cdot\vec{U}_0^* - C_D\left|\vec{v}\right|\vec{v}\cdot\vec{v} - \int_{-h-\delta}^{0}\varepsilon dz - \left[\overline{W'\left(\frac{P'}{\varrho_0}+\frac{C'^2}{2}\right)}\right]\Bigg|^{0}.\end{aligned}\right\} \tag{17}$$

This energy equation is in fact identical to the one proposed by P.R.T., if the energetics associated with $\vec{U}_0^*$, the surface vertical flux of perturbation energy, and dissipative processes are neglected, and the right-hand side is equal to zero. P.R.T. do not present a derivation of this equation, together with the potential energy conversion $\int_{-h}^{0}\overline{W'T'}\,dz$, which is consistent with their model. Deleting the aforementioned fluxes and setting $\dot{Q}_0 = 0$, P.R.T.'s Richardson Number criterion during deepening is found by eliminating $\vec{\tau}_0\cdot\vec{v}$ and $\partial T/\partial t$ from equation (17) via equations (3) and (4); viz., it is given by the balance of the last term on the left-hand side of (15) and the term on the right-hand side of equation (15),

$$-\frac{1}{2}\frac{\partial h}{\partial t}\left[\vec{v}\cdot\vec{v} - \frac{\alpha g h}{\varrho_0}(T+\Gamma h)\right] = 0. \tag{18}$$

P.R.T. define a Richardson Number as $R_i \equiv \alpha g h(T+\Gamma h)/\varrho_0 v^2$, and note that initial deepening of the layer is governed by setting the term in the brackets on the left-hand side equal to zero. Their Richardson Number criterion $R_i = 1$ is governed by equating perturbation K.E. production at the layer base to increase of P.E. within the water column. Physically, such a balance leads to layer deepening only if the shear of the mean flow across the layer bottom is increasing with time. The assumption of such a balance cannot hold after half an inertial cycle. Observations tell us, nevertheless, that mixed layers continue to deepen during a storm. The resolution to this problem is contained in a consideration of the energy equations for mean motions and perturbations, separately. The treatment of the *total energy* equation (17) as carried out by P.R.T. is misleading, for it is not useful to compare the magnitude of the separate terms on either side of the equation. The total energy equation is the sum of two linearly independent equations, whence large terms can cancel identically against other large terms. P.R.T. note that the rate of work done by the stress on the mean flow (which has no shear) is much larger than the rate of work done by the stress on the wind-wave driven flow (with very weak vertical shear), i.e., $\vec{v} \cdot \vec{\tau}_0 \gg \vec{U}_0^* \cdot \vec{\tau}_0$. However, the latter term is to be compared with other terms in the *perturbation kinetic energy balance* in equation (15) or (18), i.e., $\vec{\tau}_0 \cdot \vec{U}_0^*$ compared with $\frac{1}{2} \vec{v} \cdot \vec{v}\, \partial h/\partial t$. When such a comparison is made, and the full non-linear model is explored, it can be shown that the balance which was postulated by P.R.T. cannot hold. While an initial rapid deepening to the final slow erosion does happen, the energetics of this process are different than postulated by P.R.T. In oceanic parameter ranges of m_0 (Section 5), τ_0, and Γ, there is, however, a quantitative similarity to the actual full solutions (with finite m_0) and the ones proposed by P.R.T.

5. Some simple solutions to mixed layer formation

During deepening, the most useful variables for integration are as follows:

$$\left.\begin{aligned} \vec{U} &= h\vec{v} && \text{- Ekman transport,} \\ \theta &= \varrho_0 C_p \left(T + \frac{\Gamma h}{2}\right) h && \text{- heat content,} \\ h & && \text{- depth of the mixed layer.} \end{aligned}\right\} \tag{19}$$

Furthermore, let the energy flux from the surface input, less the amount used in dissipation, be given by

$$\overline{-W'\left(\frac{P'}{\varrho_0} + \frac{C'^2}{2}\right)}\Bigg|^0 + \tau_0 \cdot U_0^* - \int_{-h}^{0} \varepsilon dz = m_0 U_0^{*3} = m_0 \left|\frac{\vec{\tau}_0}{\varrho_0}\right|^{3/2}. \tag{20}$$

In the notation used by Denman (1973) and Kraus and Turner (1967), $m = (C_{10}\varrho_a/\varrho_0)^{1/2} m_0$, where C_{10} is the drag coefficient of the atmospheric wind profile at 10 meters and ϱ_a is the density of air; m_0 is an order-one quantity, while $m \sim 10^{-3}$. Equations (3), (4), and (15) are now written as

$$\frac{\partial \vec{U}}{\partial t} + \vec{f} \times \vec{U} = \frac{\vec{\tau}_0}{\varrho_0} - \frac{C_D}{h^2}\left|\vec{U}\right| \vec{U}, \tag{21}$$

$$\frac{\partial \theta}{\partial t} = \dot{Q}_0, \tag{22}$$

$$\frac{\partial h}{\partial t}\left[\frac{\alpha g}{C_p \varrho_0^2}\theta h^2 + N^2 \frac{h^4}{2} - \vec{U} \cdot \vec{U}\right] = 2 m_0 \left|\frac{\vec{\tau}_0}{\varrho_0}\right|^{3/2} h^2 - \frac{\alpha g}{C_p \varrho_0^2} \dot{Q}_0 h^3. \tag{23}$$

In equation (23), $N = (\alpha g/\varrho_0 \Gamma)^{1/2}$, the Väisälä frequency of the ocean below the mixed layer.

a. Initial deepening without heating, $\dot{Q}_0 = 0$, $C_D = 0$. The solutions of interest are that at time $t = 0$, $\vec{U}, T, h = 0$, and at $t = t^+$, a wind of constant strength starts to blow in the X direction.

The solution to (21) is

$$\vec{U} = \frac{\tau_0}{f\varrho_0}[\sin(ft), \quad \cos(ft) - 1]. \tag{24}$$

Since $\theta(t = 0) = 0$ and $\dot{Q}_0 = 0$, $\theta(t) = 0$, and the equation for h is

$$\frac{\partial h}{\partial t}\left[\frac{N^2h^4}{2} - \left(\frac{\tau_0}{f\varrho_0}\right)^2 2(1 - \cos ft)\right] = 2m_0\left(\frac{\tau_0}{\varrho_0}\right)^{3/2} h^2. \tag{25}$$

The solution at initial time, $t < N^{-1} < f^{-1}$, is found by examining of the asymptotics of (25). The problem is nonlinear, and considerably different behavior is expected when m_0 is finite from the case in which $m_0 = 0$. For finite t, let

$$h \sim at^{k_0} + bt^{k_1} + \ldots, \tag{26}$$

whence the leading contribution to equation (25) for small t is obtained from

$$k_0 a t^{k_0 - 1}\left[\frac{N^2}{2}a^4t^{4k_0} - \left(\frac{\tau_0}{\varrho_0}\right)^2 t^2\right] = 2m_0\left(\frac{\tau_0}{\varrho_0}\right)^{3/2} a^2t^{2k_0} + \ldots. \tag{27}$$

It is seen that as $t \to 0$, the balance must either be between the first term in the brackets on the left-hand side of the equation and the term on the right-hand side, or between the terms in the brackets. The latter choice gives $k_0 = \frac{1}{2}$, however the term proportional to m_0 is larger than that retained. Thus, $k_0 = 1/3$, $k_1 = 1$, and it is a simple matter to show that

$$h \sim (12m_0)^{1/3}\left(\frac{\tau_0}{\varrho_0 N^2}\right)^{1/2}(tN)^{1/3} + \frac{1}{30m_0}\left(\frac{\tau_0}{\varrho_0 N^2}\right)^{1/2}(tN) + \ldots. \tag{28}$$

The constant $m_0 \sim 0(1)$, whence the first term on the right-hand side of (28) is a good approximation (10% error) to the solution for $tN < 50$. P.R.T. set $m_0 = 0$, equated the terms in the brackets on the left-hand side of (25), and postulated that during the initial development, the layer depth would be given by (P.R.T. eq. (4.2))

$$h \sim 2^{1/4}\left(\frac{\tau_0}{\varrho_0 N^2}\right)^{1/2}(tN)^{1/2}. \tag{29}$$

The source of perturbation kinetic energy in P.R.T.'s solution is from the shear of the accelerated mean flow at the base of the mixed layer. With actual surface stirring and shear, during the initial development of the layer, a source of kinetic perturbation energy is available in the surface, U_0^*, layer at a constant rate the instant that the wind turns on. The perturbation kinetic energy of the mean flow is proportional to $\partial h/\partial t$ and becomes available as the layer actually starts deepening. Hence, in a physical situation, the deepening until $tN < 50$ is expected to follow expansion (28) rather than (29).

Experiments (Kato and Phillips, 1969) in an annular tank of stratified brine which was driven at the surface with a rotating grid, obtain a 30% effective fit to data of mixed layer deepening as $h \sim (15)^{1/3}\left(\frac{\tau}{\varrho_0 N^2}\right)^{1/2}(tN)^{1/3}$ for the initial deepening of the grid-mixed layer, whence $m_0 = (15/12)$, $m = 1.65\times10^{-3}$. (Denman and Miyake, 1973, use $m = 1.2\times10^{-3}$). Moore and Long (1971) report a similar finding, where mixing is done by air jets. The excellent agreement of P.R.T.'s calculation in equation (29) with Kato and Phillips' (1969) experiments are perhaps best explained in the scaling for h, rather than the (Nt) power law; in Kato and Phillips' experiments, $10 < (Nt) < 1000$, whence no single power-law in (Nt) can describe the behavor adequately, while a single scaling in amplitude is appropriate. The perturbation kinetic equation used by P.R.T. was not complete for the discussion of Kato and Phillips' experiments. The singular nature of the nonlinear equation for vanishing m_0 is apparent. The class of analytical solutions discussed by P.R.T. is a singular, special solution to the equations and is not a solution for any finite m_0 (or Q), however small.

b. Solutions with constant wind and heating. To study the nature of the deepening over a period of many pendulum days, equations (21)–(23) are cast into a nondimensional form. Although it was shown in (a) that initial deepening occurs rapidly on a time scale of N^{-1}, f^{-1} scale governs the development of the Ekman transports, whence f^{-1} will be the time scale for numerical integration and $(\tau_0/\varrho_0 f)$ will be the scale for the Ekman transport. The scale for the layer depth will be $\tau_0^{1/2}(\varrho_0 Nf)^{-1/2}$ with τ_0 the maximum wind stress, as suggested by P.R.T., and the scale for the heat content will be $(\dot{Q}_0/f)$, where $\dot{Q}_0$ is the maximum value of the heating rate.

The nondimensional equations are

$$\frac{\partial \vec{U}}{\partial t} + \hat{k}\times\vec{U} = \vec{\tau} - \frac{C}{h^2}\left|\vec{U}\right|\vec{U}, \tag{30}$$

$$\frac{\partial \theta}{\partial t} = \dot{Q}, \tag{31}$$

$$\frac{\partial h}{\partial t}\left[B\theta h^2 + \frac{h^4}{2} - \vec{U}\cdot\vec{U}\right] = A\,|\tau|^{3/2}h^2 - B\dot{Q}h^3. \tag{32}$$

Two small parameters appear: $A = 2m_0(f/N)^{1/2} < 1$ and $B = (\dot{Q}_0 N/C_p\tau_0\Gamma) < 1$. Parameter A governs the behavior of the solution for h initially and for large time $h_\infty \to (A/B)$, provided $\left|\vec{\tau}\right|$ and $\dot{Q}$ remain constant and $h(t = 0) < h_\infty$. The damping

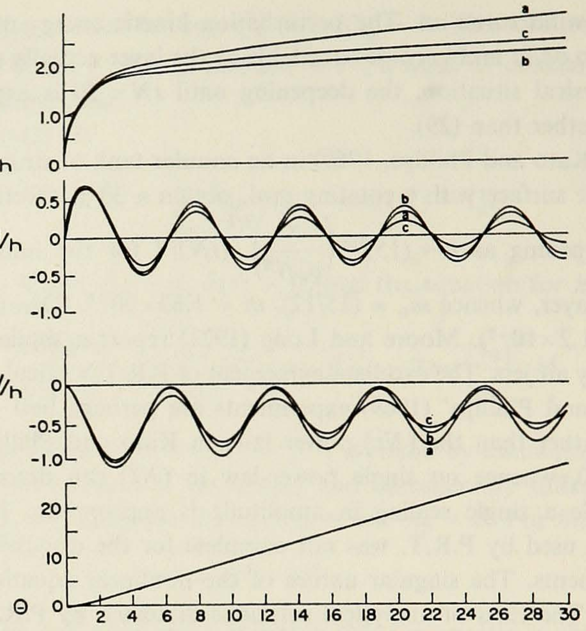

Figure 2. Solutions for the depth of the mixed layer, horizontal inertial motions, and temperature at onset of a strong wind. The parameters are given in the text; the time 2π corresponds to one pendulum day.

coefficient for the inertial motions is $C = (N/f)\, C_D < 1$ and essentially reduces the value of $\vec{U} \cdot \vec{U} \rightarrow 1 + 0(C)$ after many inertial cycles.

The solutions to these equations have been developed numerically over a range of parameters indicative of open-ocean conditions. If the initial value of $h < 2$, a rapid initial deepening of the layer within half a pendulum day to $h \simeq 2$ (or $2\,(\tau_0/\varrho_0 Nf)^{1/2}$ in dimensional units). A persistent, slow erosion follows the rapid initial deepening. In Figures 2 and 3, solutions for h, U, V, and θ are displayed for a variety of initial values, $h(0)$, and it is seen that after a few pendulum hours the solutions become independent of $h(0)$. Secondly, while a strong inertial signal is apparent in the horizontal velocity components, erosion of the thermocline is monotonic and inertial oscillations in the mixed layer depth are imperceptible. Figure 2 displays the solutions for parameters (a) $A = .100$, $B = 0$, $C = .02$; (b) $A = .100$, $B = .0333$, $C = .02$; and (c) $A = .250$, $B = .0833$, $C = .1$. Note that the rapid initial deepening appears to be relatively independent of these parameters in the limits used here; the persistent slower erosion process is retarded when the column is heated ($B > 0$), and the rate of erosion is a function of the ratio (A/B). In cases presented here, this ratio has a constant value. With constant heating and wind stress, the layer depth approaches (A/B) in a time scale which is a complicated function of A, B; however, this process was not culminated in the solutions here considered. If $B = 0$, h increases without bound as $(6\,At)^{1/3}$. This latter solution is identical to the solution as $t \rightarrow 0$, whence

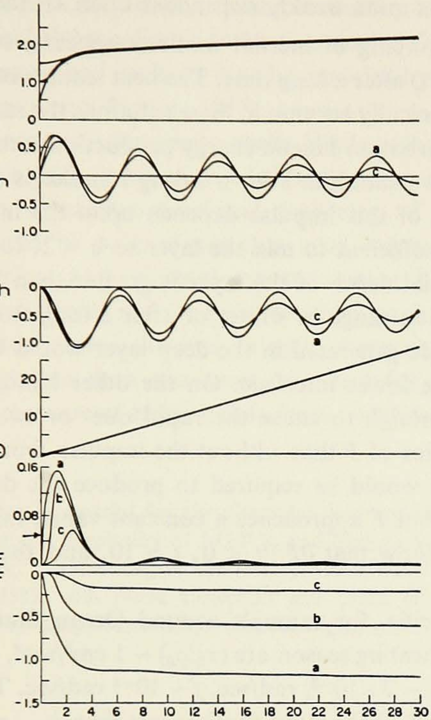

Figure 3. The effect of initial mixed layer depth on production rates of turbulent energy. The parameters and definitions of variables are given in the text; the time 2π corresponds to one pendulum day.

it is seen that inertial motions modify the layer development during an intermediate time span. For a more succinct picture of how the impulse of turbulent energy is supplied to the mixed layer by the inertial motions, consider Figure 3 in detail.

Figure 3 presents three solutions with initial values of $h(0) = 0.5$, $C = .02$ (a) 1.0, $C = .02$ (b); 1.5, $C = .1$ (c). The initial conditions of $\vec{U}(0) = 0$ are carried over, and a constant wind and heating are applied at $t = 0$: $A = .100$, $B = .0333$, $C = .02$ or $.10$. The function $E \equiv {}^1/_2\, \vec{v} \cdot \vec{v}\, \partial h/\partial t$ is the rate at which turbulent energy is released by the wind-driven inertial motions at the base of the mixed layer; for comparison, the constant production rate of turbulent energy by wave-driven turbulence is .05 (or ${}^1/_2 A$ in nondimensional units) and is marked by an arrow. Solutions were also computed for an h^*, the layer depth which would result without mean-motion energetics; these are the solutions discussed by Denman (1973) and Kraus and Turner (1967). The function, $F = h^* - h$, is the difference between mixed-layer depth without and with mean-motion energetics; it is also displayed in Figure 3. The following patterns emerge:

1. Inertial motions are only weakly dependent upon initial depth of the layer,

and the layer erosion is quite weakly dependent upon C; the larger the value of C, the more rapid the damping of inertial motions appears, such that $\vec{U} \cdot \vec{U} \gtrsim 1$ (the steady Ekman transport) after a long time. The heat storage of the layer, here denoted by θ, increases monotonically because in these solutions the rate of heating is constant.

2. Initially, the peıturbation kinetic energy production rate is governed by surface processes, and within a pendulum hour a strong impulse is received from the mean motions. The strength of this impulse depends upon the initial depth of the layer and is seen to be just sufficient to mix the layer to h ~ 2, for typical oceanic values of A and B. If the initial depth of the layer is greater than 2, as would be the case in the cooling convection range in winter or after a long stormy period in summer, mean motions which are generated in the deep layer would be too weak to produce turbulent energy at the lower interface. On the other hand, the surface layer production is not large enough to cause the rapid one- or two-day deepening, and it is seen from the behavior of F that without the impulse from the accelerated mean flow, a very long time would be required to produce the deepening to $h = 2$. On the graph, it appears that F approaches a constant value; however, examinations of the numerical output show that $\partial F/\partial t < 0$, $t > 10$, and the solutions for h and h^* will eventually merge.

3. In the North Pacific, for example around Ocean Station PAPA, the typical parameters during the heating season are $(\tau_0/\varrho_0) \sim 1$ cm²/sec², $\dot{Q} = 1 \times 10^{-2}$ cal/cm²sec, $\Gamma \sim 5 \times 10^{-4}$ C'°/cm, $N \sim 3 \times 10^{-3}$, rad/sec, $f \sim 10^{-4}$ rad/sec. The parameters $A \simeq 2 \sim 3 \times 10^{-1}$, $B \simeq 5 \sim 8 \times 10^{-2}$. If the layer depth is less than $h_{min} \simeq 25 \sim 30$ m, very rapid deepening would follow to this depth. Note that calculations of Denman and Miyake (1973) at PAPA began with $h(0) > h_{min}$, whence in this context it represents a slow erosion by surface processes. On the other hand, Turner's (1969) interpretation of rapid half-pendulum-day deepening of the mixed layer, from observations near Bermuda (Stommel, *et al.*, 1969), can now be cast in a new light in terms of the mean-motion production rate. Turner found that surface production, which was consistent with laboratory experiments and the later results of Denman and Miyake (1973), could not account for the deepening of the layer from 15 m to 40 m in a few hours. As $h(0) < h_{min}$ in this case, inertial motions presumably supplied the mixing energy on a time scale of a few hours. One of the central points of this calculation is that an impulse of the wind can start strong inertial motions and can lead to a large-scale increase in the available turbulent energy and to a rapid deepening. The mechanism for deepening and the rate of subsequent erosion with the onset of strong winds critically depend upon the depth of the mixed layer at the onset of the winds.

In this presentation, we have not explored the more general problem of forcing by variable atmospheric inputs; these are most useful when direct comparison with ocean-station data is made. For example, it is apparent that during slack wind conditions in the heating season, the deepening is arrested abruptly (the right-hand side of equation (23) vanishes) and presumably a new, warm, well-mixed layer will begin to erode into the existing homogeneous column. Secondly, the treatment of dissipation in these quasi-periodic solutions becomes much more crucial. For example, if a build-up of a seasonal thermocline is to be constructed from a sequence of progressively warm mixed layers and its decay is to be modeled by penetrative convection by this model (vide Kraus and Turner, 1967), a net amount of potential energy is put into the water column over the yearly cycle. While seasonal trends can always be computed for a single year, integration of this model over many years, without dissipation, will lead to unrealistic, continual deepening of the entire mixed-layer column. Finally, it should be pointed out that near coasts, in western boundary currents, and in the tropics, the horizontal convergences and divergences are strong and the mixed-layer development is strongly affected by an interaction with the deep ocean and adjacent water masses. As was so aptly demonstrated by Denman and Miyake (1973), ocean station data can serve as a test for the single-dimensional models; this author is not aware of existing data from which tests of nonlocal theories can be made.

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REFERENCES

Denman, K. L. 1973. A time-dependent model of the upper ocean. J. Phys. Oceanogr., *3*: 173–184.

Denman, K. L., and M. Miyake. 1973. Upper layer modification at ocean station PAPA: Observations and simulation. J. Phys. Oceanogr., *3*: 185–196.

Ekman, V. Å. 1905. On the influence of earth's rotation on ocean currents. Arkiv. Mat. Astron. Fysik., *2*: 1–53.

Geisler, J. E., and E. B. Kraus. 1969. The well-mixed Ekman boundary layer. Deep-sea Res., Suppl., *16*, 73–84.

Grant, H. L., A. Moilliet, and W. M. Vogel. 1965. Some observations of the occurrence of turbulence in and above the thermocline. J. Fluid Mech., *34*: 443–448.

Kato, H., and O. M. Phillips. 1969. On the penetration of a turbulent layer into stratified fluid. J. Fluid Mech., *37*: 643–655.

Kraus, E. B., and J. S. Turner. 1967. A one-dimensional model of the seasonal thermocline, Part II. Tellus, *19*: 98–105.

Miropolskiy, Yu. Z. 1970. Nonstationary model of the wind-convection mixing layer in the ocean. Izv., Atm. and Ocean. Phys., *6*(12): 1284–1294.

Moore, M. J., and R. R. Long. 1971. An experimental investigation of turbulent stratified shearing flow. J. Fluid Mech., *49*: 635–655.

Phillips, O. M. 1966. The Dynamics of the Upper Ocean. Cambridge University Press, 264 pp.

Pollard, R. T., and R. C. Millard, Jr. 1970. Comparison between observed and simulated wind-generated inertial oscillations. Deep-sea Res., *17*: 813–821.

Pollard, R. T., P. B. Rhines, and R. O. R. Y. Thompson. 1973. The deepening of the wind-mixed layer. J. Geophys. Fluid Dyn., *3*: 381–404.

Pond, S. 1972. The exchanges of momentum, heat and moisture at the ocean-atmospheric interface. Proc. Nat. Acad. Sci., Conf. on Numerical Models of Ocean Circulation, Oct., 1972, Durham, N.H.

Stommel, H., K. Saunders, W. Simmons, and J. Cooper. 1969. Observations of the diurnal thermoclibe. Deep-sea Res., *16* Suppl., 269–284.

Turner, J. S. 1969. A note on wind-mixing at the seasonal thermocline. Deep-sea Res., *16*: Suppl., 297–300.

Warren, B. 1972. Insensitivity of subtropical mode water characteristics to meteorological fluctuations. Deep-sea Res., *19*: 1–19.

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