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ON THE MUTUAL ADJUSTMENT OF PRESSURE AND VELOCITY DISTRIBUTIONS IN CERTAIN SIMPLE CURRENT SYSTEMS*

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The writer (*1*) recently advanced the hypothesis that the horizontal pressure gradients observed in the current systems of the atmosphere and the ocean to a large extent must be interpreted as reactions to the Coriolis' forces impressed upon these systems by the rotation of the earth. Observational support for this point of view was found in the fact that both the temperature-salinity and oxygen-salinity correlation curves obtained from stations on both sides of the Gulf Stream in the region between Nova Scotia and Bermuda are very nearly identical, even though the individual isotherms may drop as much as 700 meters between the slope water basin and the Sargasso Sea. This mass distribution and the resulting pressure distribution are most readily interpreted as the result of a continuous banking process caused by the action of the Coriolis' forces on the moving masses of the Gulf Stream system. Theoretical support was found in a theorem by Taylor (*2*), according to which any purely two-dimensional motion which can occur in a non-rotating system is equally possible also when the system rotates at uniform speed around an axis normal to the plane of motion, the deflecting forces due to the rotation then being offset by the reaction of the fluid system in the form of a superimposed "Coriolian" pressure field. It is easily shown that this nullification of the Coriolis' forces is possible only when the prescribed motion is purely two-dimensional.

Both in the atmosphere and in the ocean balancing pressure gradients are established through the piling up of mass in certain regions and the removal of mass from others, i. e. through vertical motions. Thus the conditions upon which the complete validity of Taylor's theorem depends are not

* This note constitutes a preliminary report on certain theoretical investigations now in progress at the Massachusetts Institute of Technology. Other phases of the problem discussed below are being investigated by Messrs. J. Holmboe, G. Grimminger and H. Wexler. The author is greatly indebted to Mr. Grimminger for all the numerical computations in connection with the evaluation of the functions F and Φ referred to below and for the preparation of the various diagrams. Mr. Grimminger's computations were in part carried out with the aid of the differential analyzer belonging to the Electrical Engineering Department of the Massachusetts Institute of Technology; the assistance of the staff members in charge is gratefully acknowledged.

satisfied during the time required for the establishment of the "Coriolian" pressure field. For this reason it seems desirable to study, in some detail, the process whereby dynamic balance is created and to determine if there are any limitations to the ability of a given system to build up a Coriolian pressure distribution. The existence of such limitations would necessitate modifications in the current theory previously suggested by the author on the basis of Taylor's theorem.

There are some indications that such limitations exist. Experiments with wake streams in rotating tanks carried out by Mr. Spilhaus (*3*) and described elsewhere in this issue suggest that complete balance between pressure gradient and deflecting force is never reached. In all cases examined by Mr. Spilhaus the current pattern indicates an inadequate development of the Coriolian pressure field, resulting in anticyclonic current trajectories resembling inertia circles.

Recently attempts have been made at the Massachusetts Institute of Technology to determine trajectories in the atmosphere through a study of the movements of tongues of moist and dry air within selected isentropic surfaces (*4*). The daily charts constructed for this purpose show that individual air currents have a pronounced tendency to break up into large anticyclonic eddies, presumably as a result of inadequate development of the Coriolian pressure gradients.

To the author's knowledge no theory exists which satisfactorily describes the mechanism whereby the mass and pressure distributions adjust themselves to the velocity distribution, although the problem is of great practical importance; its solution would be of value not only to physical oceanographers but also to meteorologists in connection with the interpretation of the so-called dynamic pressure formations (i. e. warm anticyclones, cold cyclones). The problem is now being analyzed in some detail by the author and his collaborators. The purpose of the present article is merely to call attention to the important rôle played by lateral (isentropic) mixing in this process.

To illustrate the mechanism of adjustment it is advisable to analyze first the changes which must occur in a straight parallel current encircling the earth at a given latitude and characterized by complete symmetry with respect to the axis of the earth. A somewhat simpler system may be obtained by considering the effect of lateral diffusion on a linear current system in a rotating plane. It is in this form that the problem will be studied below.

We shall first describe the lateral diffusion of momentum in a parallel current in a homogeneous, incompressible and non-rotating medium of constant density ρ. A mathematical theory for such a current system is easily obtained through an adaptation of Tollmien's experimentally well established theory for two dimensional wake streams (*5*). Fig. 5 demonstrates three successive velocity profiles resulting from lateral diffusion within such a system. If one designates the velocity by u and measures the horizontal y-coordinates to the left from the down-stream axis of the current, the equation of motion reduces to

$$\rho D \frac{\partial u}{\partial t} = \frac{\partial \tau D}{\partial y}, \tag{1}$$

where t is the time and D the depth of the fluid, which in this case remains constant from point to point and throughout the entire process. There

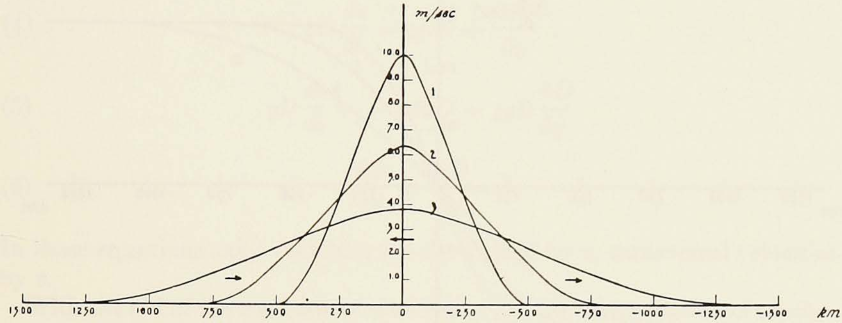

Figure 5. Successive velocity profiles in stationary system. Arrows indicate direction of transversal displacements which would result from the action of a superimposed Coriolis force.

are no transversal velocity components. The shear stress τ vanishes in the resting fluid surrounding the current and thus

$$\rho D \int_{-\infty}^{+\infty} u dy = \text{constant}. \tag{2}$$

The integral (2) measures the momentum of the current and remains constant throughout the diffusion process. Thus the areas under the various curves in Fig. 5 are equal.

Proceeding now to the diffusion of momentum in a rotating system it is evident that equation (1) and the velocity profiles in Fig. 5 should represent the results of the diffusion process also in a rotating system, provided transversal displacements do not appear and provided the shearing stresses are unaffected by the rotation. If it be assumed that the time rate of diffusion is so slow that the current at each instant remains in dynamic equilibrium (balance between pressure gradient and deflecting force), successive profiles of the free surface should then have the shape indicated in Fig. 6. These curves have been computed from the curves in Fig. 5 with the aid of the equation for gradient motion,

$$u = -\frac{g}{f}\frac{\partial D}{\partial y}, \tag{3}$$

in which expression g represents the acceleration of gravity and f the Coriolis' parameter.

An inspection of Fig. 6 reveals that the change in pressure distribution illustrated by these curves cannot be accomplished without some transversal transfer of mass (northward, in case the current moves eastward). As soon as such transversal displacements occur, equation (1) no longer ex-

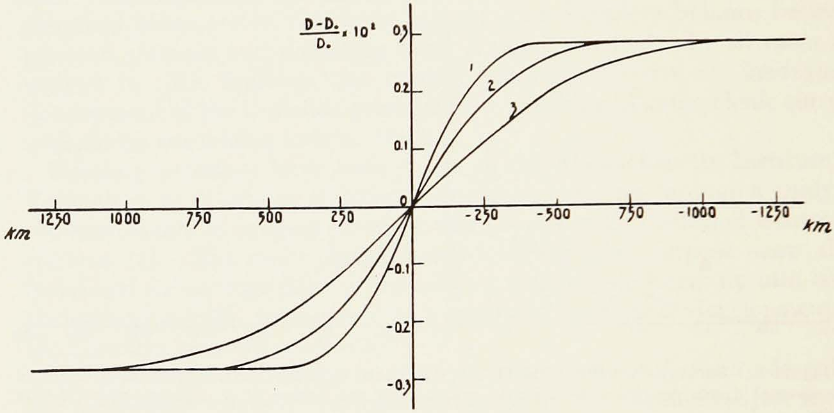

Figure 6. Successive free surface profiles corresponding to the velocity profiles in figure 5, assuming at all times equilibrium between Coriolis' force and pressure gradient. The curves indicate that the gradual adjustment of the free surface can not take place without transfer of mass from right to left across the current. The value of D_0 used in computing these curves from the ones in Figure 5 corresponds to the height of the homogeneous atmosphere (8 km.).

presses the true balance of forces. Without resort to formulae, the general character of the diffusion process may be described as follows:

Through lateral mixing the layers immediately left and right of the main current will receive a slight acceleration down-stream while the center portion will lose in speed. Thus the immediate effect is to establish velocities in excess of the gradient value to the left and right of the current axis and of less than the gradient value in the axis itself. The resulting unbalance produces slight transversal velocities towards the left in the center of the current while the immediate environment, characterized by supergradient velocities, tends to move towards the right. These transversal components, which are indicated by the arrows in Fig. 15, must produce convergence with pressure rise in the left half of the current system, divergence with pressure drop in the right; the mass distribution is thus gradually adjusted towards the new velocity distribution.

Since the transversal velocities must vanish at great distances from the axis it is evident that the fluid portions far to the left of the current must experience divergence, those far to the right convergence. Thus the current is gradually set off from its environment by a trough of low pressure to the left, a ridge of high pressure to the right.

A quantitative estimate of the pressure changes described above may be obtained with the aid of the momentum integral. The equations of motion and the equation of continuity for the linear current system here described may be written in the following form:

$$\rho D \frac{du}{dt} = \rho f v D + \frac{\partial \tau D}{\partial y} \tag{4}$$

$$\rho D \frac{dv}{dt} = - \rho f u D - \rho g D \frac{\partial D}{\partial y} \tag{5}$$

$$\frac{\partial D}{\partial t} + \frac{\partial v D}{\partial y} = 0 \tag{6}$$

In these equations axial velocities are designated by u, transversal velocities by v.

With the aid of the equation of continuity (6), the first equation of motion may be written in the form

$$\frac{\partial}{\partial t}[\rho D(u - fy)] + \frac{\partial}{\partial y}[\rho v D(u - fy)] = \frac{\partial \tau D}{\partial y}. \tag{7}$$

Integration across the current system into the undisturbed surrounding regions gives

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial t}[\rho D(u - fy)]\, dy = 0 \tag{8}$$

or

$$\frac{\partial}{dt}\int_{-\infty}^{+\infty} u D dy = f \int_{-\infty}^{+\infty} y \frac{\partial D}{\partial t} dy. \tag{9}$$

Equation (8) expresses the fact that the diffusion process leaves the total *absolute* momentum of the current system unchanged.

If the diffusion is slow it may be assumed that the current system passes through a series of successive dynamic equilibrium states; consequently the second equation of motion (5) reduces to the form

$$\frac{\partial D}{\partial y} = - \frac{fu}{g}. \tag{10}$$

The integral on the left side of (9) may then be integrated, giving

(11)
$$-\frac{g}{f}\frac{\partial}{\partial t}[D_{\infty}{}^2 - D_{-\infty}{}^2] = f\int_{-\infty}^{+\infty} y\frac{\partial D}{\partial t}dy.$$

Since the depth of the undisturbed fluid to the right ($D_{-\infty}$) and to the left ($D_{+\infty}$) of the current do not change, it follows that in this case of slow diffusion

(12)
$$\int_{-\infty}^{+\infty} y\frac{\partial D}{\partial t}dy = 0$$

It is evident that the curves in fig. 6 do not satisfy this requirement. The establishment of the successive profiles in fig. 16 would require a net transfer of fluid to the left and this transfer must, under the influence of the deflecting force, lead to the development of a net axial component in excess of the gradient value.

The magnitude of this net component (δu) is readily computed. If the final boundaries of the current are given by $y = \pm a$ it may be shown that the excess velocity component has a mean value over the entire current cross section from $-a$ to $+a$ of

(13)
$$\delta u = \frac{f\int_{-a}^{+a} y(D - D_i)dy}{\int_{-a}^{+a} Ddy},$$

in which D_i represents the initial depth and $D - D_i$ thus measures the change in depth at each point. The immediate effect of such an unbalanced velocity component is to remove fluid from the left edge of the current system and to pile it up along the right edge, in such a fashion that a trough of low pressure and a ridge of high pressure are formed, as described above.

In reality the process of diffusion and the adjustment of the mass distribution must take place continuously and not in the discrete steps described in the previous analysis. To follow the continuous diffusion process it is necessary to integrate the equations of motion (4, 5, 6). This will now be done under the assumption that the diffusion process is so slow that the system passes through a series of dynamic equilibrium states. Under those conditions the equations reduce to the simpler form

(14)
$$\rho fvD + \frac{\partial \tau D}{\partial y} = 0$$

(15)
$$fu + g\frac{\partial D}{\partial y} = 0$$

(16)
$$\frac{\partial D}{\partial t} + \frac{\partial vD}{\partial y} = 0.$$

It is easily seen that the first of these equations satisfies the momentum integral

$$\int_{-\infty}^{+\infty} y \frac{\partial D}{\partial t} dy = 0, \tag{17}$$

derived previously for the case of dynamic equilibrium. From a combination of the first and the third equation it follows that

$$-\rho f \frac{\partial D}{\partial t} + \frac{\partial^2 \tau D}{\partial y^2} = 0. \tag{18}$$

This equation may be integrated provided we know the value of the shearing stress. Let ν represent the kinematic eddy viscosity characteristic of the lateral mixing process. Then

$$\frac{\tau}{\rho} = \nu \frac{\partial u}{\partial y} = -\frac{\nu g}{f} \frac{\partial^2 D}{\partial y^2}, \tag{19}$$

and consequently

$$\frac{\partial D}{\partial t} + \frac{g}{f^2} \frac{\partial^2}{\partial y^2}\left(\nu D \frac{\partial^2 D}{\partial y^2}\right) = 0. \tag{20}$$

If one introduces for νD a constant value $\nu_o D_o$*, where D_o represents the mean depth of the fluid, equation (20) reduces to the form

$$\frac{\partial D}{\partial t} + \nu_o \cdot \frac{g D_o}{f^2} \frac{\partial^4 D}{\partial y^4} = 0. \tag{21}$$

This equation may be written in a simpler form if one introduces non-dimensional time and length scales,

$$y = \lambda \eta \tag{22}$$

$$s = \frac{\nu_o t}{\lambda^2} \tag{23}$$

$$\left(\lambda = \frac{1}{f} \sqrt{g D_o}\right)$$

Then

$$\frac{\partial D}{\partial s} + \frac{\partial^4 D}{\partial \eta^4} = 0 \tag{24}$$

and consequently also

* The assumption $\nu D = \nu_o D_o$ is obviously quite satisfactory in the case of a deep homogeneous medium. In case of a stratified medium this assumption implies that within a given layer the kinematic eddy viscosity resulting from lateral mixing is inversely proportional to the depth of that particular isopycnic (in the atmosphere, isentropic) layer. This assumption is in qualitative agreement with Parr's (*6*) results concerning the relation between vertical stability and the intensity of isopycnic (isentropic) mixing.

(25) $$\frac{\partial u}{\partial s} + \frac{\partial^4 u}{\partial \eta^4} = 0.$$

This final equation is easily integrated. If one assumes that the initial velocity distribution is given by

(26) $$u_o = \psi(\eta),$$

the velocity distribution at any subsequent time s will be given by

(27) $$u = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi(\eta + \alpha\sqrt[4]{4s})\, F(\alpha)\, d\alpha,^{*}$$

the function $F(\alpha)$ being defined by the integral

(28) $$F(\alpha) = \int_0^{\infty} e^{-\frac{\lambda^4}{4}} \cos\lambda\alpha \cdot d\lambda, \quad \int_{-\infty}^{+\infty} F(\alpha)\, d\alpha = \pi.$$

As an illustration we shall compute the gradual diffusion of a current characterized initially by a total pressure drop cross-stream of the amount $\rho g\Delta$. Thus the initial depth of the fluid is $D_o - \frac{1}{2}\Delta$ to the left of the current and $D_o + \frac{1}{2}\Delta$ to the right of the current. It is not difficult to formulate the solution for the case of a finite initial width of the current. However, the initial width of the current is of little significance in the later stages of the diffusion process. For this reason, and in order to gain simplicity, it will be assumed that the initial width is infinitely small. Thus, from (27) one finds

(29) $$D = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi(\eta + \alpha\sqrt[4]{4s})\, F(\alpha)\, d\alpha,$$

where

(30) $$\psi(\eta) = D_o - \frac{\Delta}{2} \qquad \text{for } \eta > 0.$$

and

(31) $$\psi(\eta) = D_o + \frac{\Delta}{2} \qquad \text{for } \eta < 0.$$

Thus

(32) $$\psi(\eta + \alpha\sqrt[4]{4s}) = D_o - \frac{\Delta}{2} \quad \text{for } \alpha > -\frac{\eta}{\sqrt[4]{4s}}$$

and

(33) $$\psi(\eta + \alpha\sqrt[4]{4s}) = D_o + \frac{\Delta}{2} \quad \text{for } \alpha < -\frac{\eta}{\sqrt[4]{4s}}$$

* A detailed discussion of equation (25) and its solution (27) will be published later.

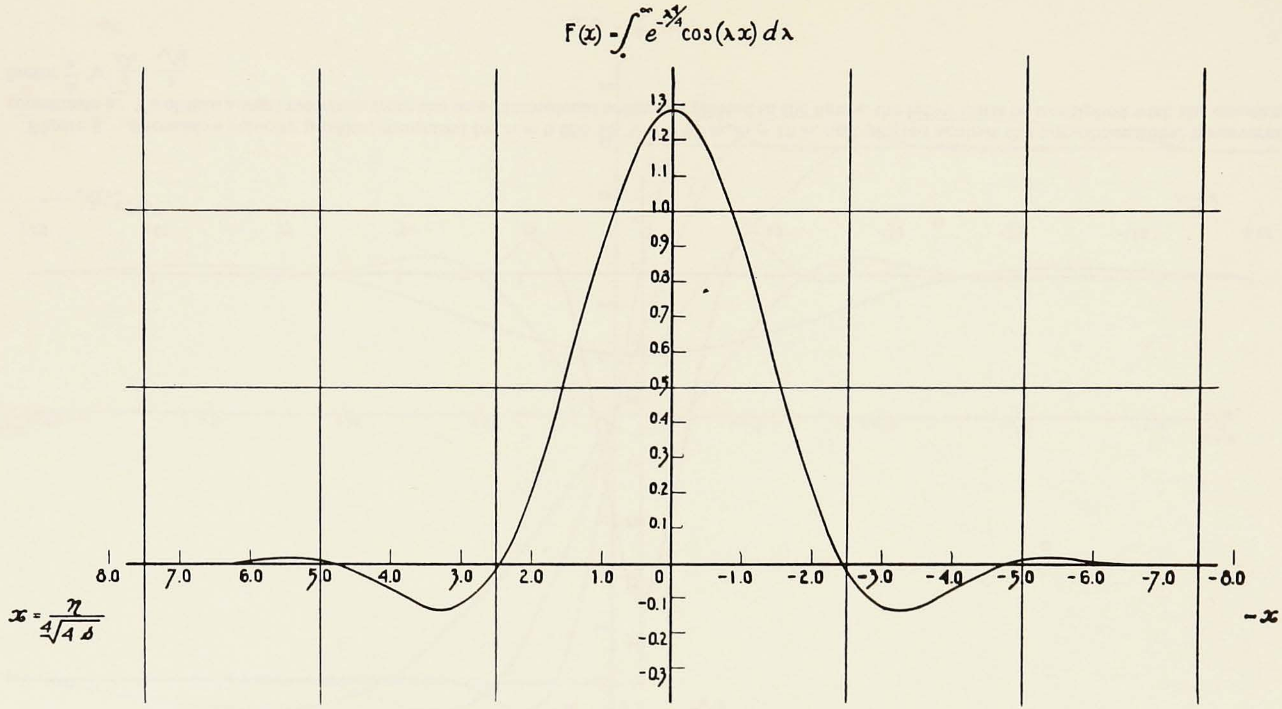

Figure 7. Non-dimensional representation of velocity distribution as a function of $\frac{\eta}{\sqrt[4]{4s}}$, taking into consideration the Coriolis' force acting upon the transversal velocity components. The ordinate is 1.2818 u/u_o, where u_o is the simultaneous velocity in the center of the current.

$$\sqrt{2}\cdot\frac{\pi}{\Delta}\sqrt{\frac{D_0}{g}}\cdot u = \frac{1}{\sqrt[4]{s}}F(x)$$

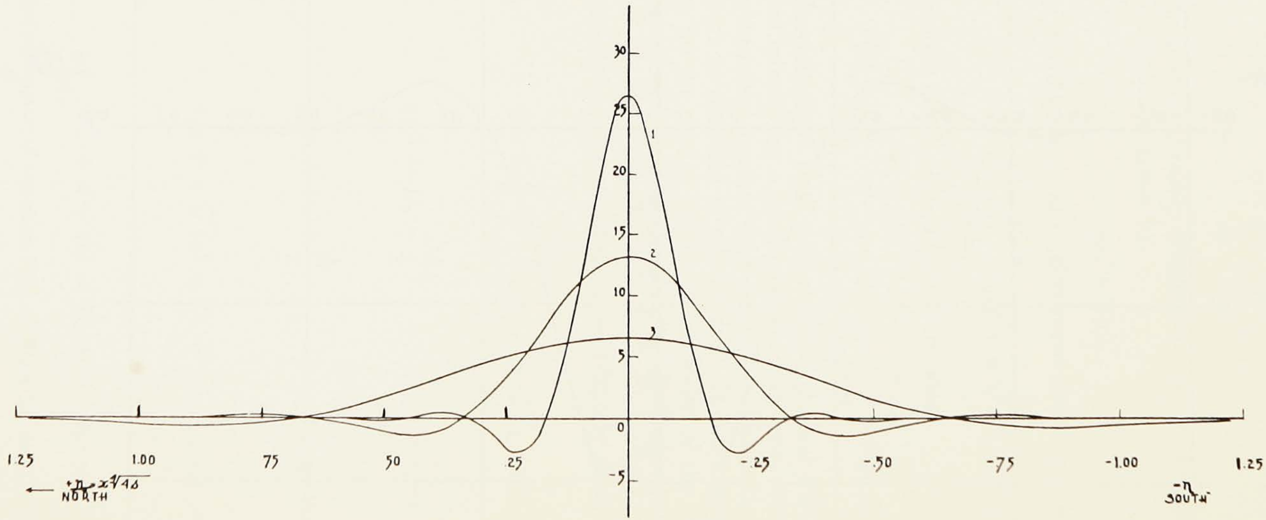

Figure 8. Successive velocity profiles, computed for $s_1 = 0.055 \cdot 10^{-4}$, $s_2 = 16\, s_1$, $s_3 = 16\, s_2$, and plotted against the non-dimensional transversal coordinate η. To obtain actual velocities from the non-dimensional ordinates plotted in the figure, the latter must be multiplied with the constant factor $\frac{\Delta}{\pi}\sqrt{\frac{g}{D_0}}\cdot\frac{1}{\sqrt{2}}$

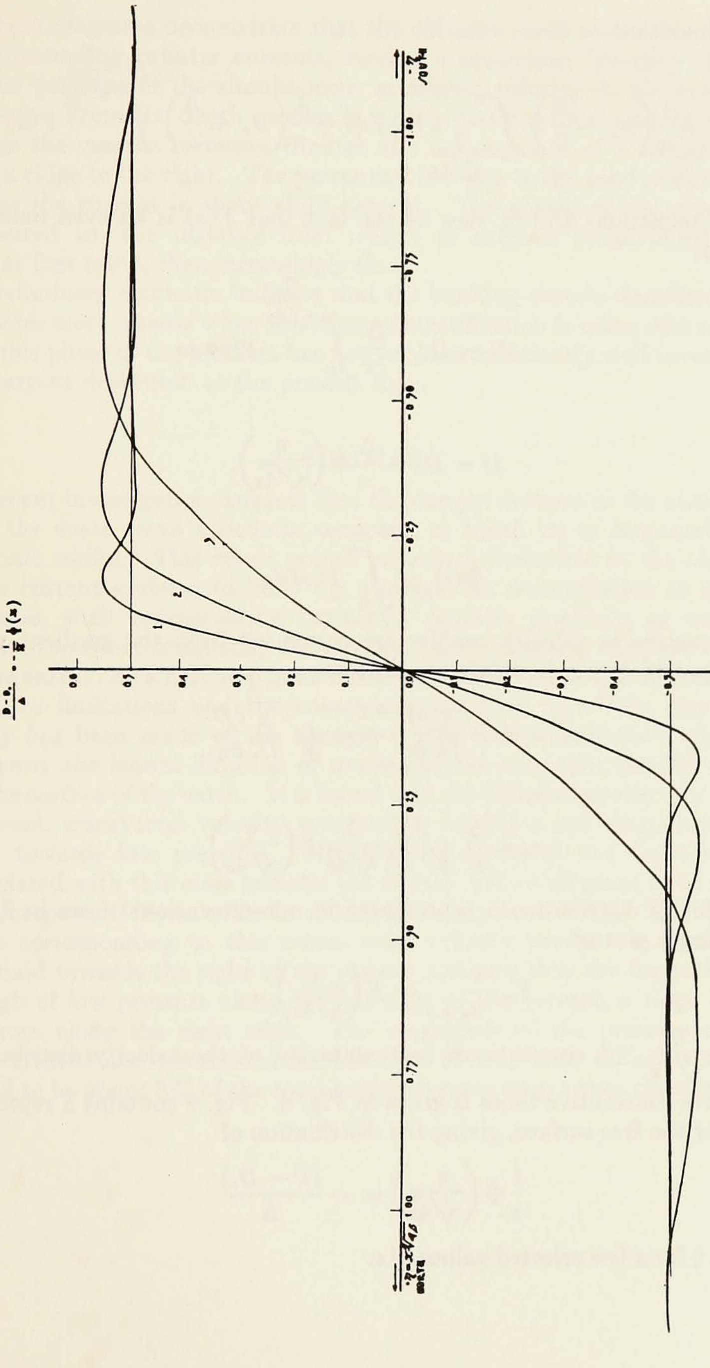

Figure 9. Successive profiles of the free surface, corresponding to the velocity profiles in Figure 8.

It follows that

$$(34)\quad D = \frac{1}{\pi}\left(D_o - \frac{\Delta}{2}\right)\int_{-\frac{\eta}{\sqrt[4]{4s}}}^{\infty} F(\alpha)\,d\alpha + \frac{1}{\pi}\left(D_o + \frac{\Delta}{2}\right)\int_{-\infty}^{-\frac{\eta}{\sqrt[4]{4s}}} F(\alpha)\,d\alpha.$$

After reductions and in view of the fact that $F(\alpha)$ is an even function one finds

$$(35)\qquad D = D_o - \frac{\Delta}{\pi}\int_0^{\frac{\eta}{\sqrt[4]{4s}}} F(\alpha)\,d\alpha$$

or

$$(36)\qquad D = D_o - \frac{\Delta}{\pi}\,\Phi\left(\frac{\eta}{\sqrt[4]{4s}}\right),$$

where

$$(37)\qquad \Phi(r) = \int_0^r F(\alpha)\,d\alpha$$

The successive velocity profiles are computed from the gradient wind equation, which may be written in the form

$$(38)\qquad u = -\frac{g}{f\lambda}\frac{\partial D}{\partial \eta} = -\sqrt{\frac{g}{D_o}}\frac{\partial D}{\partial \eta}$$

or

$$(39)\qquad u = \frac{\Delta}{\pi}\sqrt{\frac{g}{D_o}}\frac{1}{\sqrt[4]{4s}}F\left(\frac{\eta}{\sqrt[4]{4s}}\right).$$

The velocity distribution is represented in non-dimensional form in fig. 7, this being a plot of

$$(40)\qquad F_o \cdot \frac{u}{u_{\eta=o}} \equiv F\left(\frac{\eta}{\sqrt[4]{4s}}\right)$$

against $\frac{\eta}{\sqrt[4]{4s}}$. A conventional representation of the velocity distribution ψ at three consecutive times is given in Fig. 8. Fig. 9 contains a representation of the free surface, giving the distribution of

$$(41)\qquad \frac{1}{\pi}\Phi\left(\frac{\eta}{\sqrt[4]{4s}}\right) = -\frac{(D - D_o)}{\Delta}$$

against η for a few selected values of s.

These diagrams demonstrate that the diffusion leads to the development of surrounding counter currents, having a maximum intensity of about eleven per cent of the simultaneous maximum velocity in the axis of the current. From the depth profiles it is seen that the total drop in pressure across the current increases through the development of a trough to the left, a ridge to the right. The percentual increase in the total pressure drop across the current is about eight percent. The spread of the current, as measured by the distance from trough to ridge is proportional to $s^{-\frac{1}{2}}$, i. e. at first rapid, then increasingly slow.

Preliminary estimates indicate that the banking process described above becomes more intense when the effect of stratification is taken into account, but this phase of the problem has not yet been sufficiently well investigated to warrant discussion at the present time.

SUMMARY

Recent investigations suggest that the current systems of the atmosphere and the ocean have a definite tendency to break up in large-scale anticyclonic eddies. This result points towards a limitation in the ability of these current systems to build up, through the accumulation or removal of mass, such compensating horizontal pressure gradients as would be required to offset completely the Coriolis' forces resulting from the rotation of the earth. As a first step in an investigation aiming at the determination of these limitations and the consequent breaking up of the currents, a study has been made of the changes in the mass distribution which accompany the lateral diffusion of momentum in a straight parallel current on the surface of the earth. It is found that the diffusion process is attended by weak transversal velocity components having a net component from high towards low pressure. Through the action of the Coriolis' force associated with this mass transfer the current will at all times have a slight net component downstream in excess of the gradient value. The Coriolis' force corresponding to this excess axial velocity produces a banking of the fluid towards the right of the current axis and thus the formation of a trough of low pressure along the left edge of the current, a ridge of high pressure along the right edge. The magnitude of the pressure changes thus created are determined for the case of very slow diffusion and are found to be about 8% of the total initial pressure drop across the current.

REFERENCES

1. C.-G. Rossby.
1936. Dynamics of Steady Ocean Currents in the Light of Experimental Fluid Mechanics, Papers in Physical Oceanography and Meteorology, Vol. V, No. 1.

2. G. I. Taylor.
1932. The Transport of Vorticity and Heat through Fluids in Turbulent Motion, Proceedings of the Royal Society of London, Series A, Vol. 135, page 685.

3. A. F. Spilhaus.
1937. Note on the Flow of Streams in a Rotating System, Sears Found. Journal Marine Research, Vol. 1, No. 1.

4. C.-G. Rossby and collaborators.
1937. Isentropic Analysis, Bulletin of the American Meteorological Society, Vol. 18, Nos. 6–7.

5. W. Tollmien.
1926. Berechnung turbulenter Ausbreitungsvorgänge, Zeitschrift für Angewandte Mathematik und Mechanik, Band 6, p. 468.

6. A. E. Parr.
1936. On the Probable Relationship Between Vertical Stability and Lateral Mixing Processes, Journal du Conseil, Vol. XI, No. 3.